Some Arithmetic Functions in Counting Unrooted Topological Maps

and Coverings of Surfaces

Valery Liskovets

Institute of Mathematics
National Academy of Sciences
Minsk, Belarus

liskov@im.bas-net.by

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Introduction: intention

- Exact closed enumeration formulae.

- Unrooted maps and related objects, such as non-equivalent coverings of surfaces and subgroups of finitely gen. groups.

- Planar (i.e. spherical) maps and maps on other orientable surfaces.

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• A brief survey. Phenomenologically.
Selectively: initial and characteristic examples,
not necessarily latest or most general.

• Ordered by involved arithmetic functions. Two parts:
(I) classical multiplicative functions and
(II) a multivariate function introduced recently. In more detail.

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• An arithmetic function $f(n)$ is called multiplicative if $f(1) = 1$ and

$$f(km) = f(k)f(m)$$

whenever $\mathrm{GCD}(k, m) = 1$.
$f(n)$ is determined by the values $f(p^a)$ for all prime p and $a \geq 1$.

Enumeration: unrooted/unlabeled

Unrooted maps/unlabeled objects.
Typical way: enumeration in terms of rooted/labeled ones.
Burnside/Redfield/Pólya. Recall:

Burnside's Lemma (the name is improper, but. . .).

$$|G\backslash\backslash Y| = \frac{1}{|G|}\Big(|Y| + \sum_{g\in G,\, g\neq 1} |Y_g|\Big)$$

$G\backslash\backslash Y$: the set of orbits ("unlabelled" objects)
of a finite group G in its action on a set Y.

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• Reduction to separate elements of the group.

• Particularly efficient when the group G acts semi-regularly*.
Just for maps, coverings,. . . [* cycles of equal length in every $g \in G$]

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• Rooted maps (equivalent to 'labeled' ones in terms of enumerative combinatorics). Much easier to enumerate.

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$$A'(n) = \frac{2 \cdot 3^n (2n)!}{n!\,(n+2)!}$$

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• Related rooted objects (other than maps) are enumerated effectively as well, although often with not so simple formulae. Diverse techniques.

I. CLASSICAL MULTIPLICATIVE FUNCTIONS

Euler totient function: necklaces, plane trees, etc.

$\phi(n)$:=#(units of C_n). $\phi(n) = n \prod_{p|n \text{ prime}} (1 - p^{-1})$.

- As a multiplicative function:

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• Trivially arises for counting objects up to the action of cyclic groups C_n. E.g.,

$L_k(n)$:=#(necklaces with n beads of k types up to rotations).

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- Plane trees. Similarly. Less trivial.

- Chord diagrams. Similarly. Even less trivial in general. . .

Euler totient function: unrooted arbitrary planar maps

$A^+(n)$:=#(unrooted planar maps with n edges). Sense-preserving: $^+$.

Theorem [VL, 1981].

$$A^+(n)=\frac{1}{2n}\left[A'(n)+\sum_{\substack{m<n\\m|n}}\phi\left(\frac{n}{m}\right)\binom{m+2}{2}A'(m)\right]+\begin{cases}\frac{n+3}{4}A'\left(\frac{n-1}{2}\right), & n \text{ odd}\\ \frac{n-1}{4}A'\left(\frac{n-2}{2}\right), & n \text{ even}\end{cases}$$

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- A technique based on quotient maps (= orbifolds) with respect to rotations of order ℓ: spheres with two ℓ-poles.
$\ell = 2$: boring technical complications (one or two half-edges).
- Reproved and generalized later. Rather elementary presently.

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$B^+(n)$:=#(unrooted non-separable (= 2-connected) planar maps).

Theorem [VL–T.Walsh, 1983].

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• Further. Eulerian planar maps, loopless,. . . : similar formulae. Again: despite that quotient maps do not generally preserve the underlying property (even valency,. . .). Partially explained.

Möbius function: subgroups of the free group

$\mu(n)$: the Möbius inversion. (Recall: $\mu(n)$ is the multiplicative function determined by $\mu(p) := -1,\ \mu(p^a) := 0$ for p prime and $a > 1$.)

F_r:= the free group of rank $r \geq 2$,

$N_{F_r}(n)$:=#(conjugacy classes of subgroups of index n in F_r). Also:

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$$\boxed{N_{F_r}(n) = \frac{1}{n}\sum_{m|n} M_{F_r}(m) \sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(r-1)m+1}}$$

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- Again: unexpectedly simple formula. Reproved subsequently: R.Stanley, G.Jones,... Admits a smarter representation: later.

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• Proof: initially heavy and rather artificial. Presently this is a particular case of a much more general and clear result!

Coverings vs subgroups

Recall:
For connected coverings of a manifold $\mathcal{M}$
with the fundamental group $\pi_1(\mathcal{M})$

• its pointed (rooted) n-fold coverings bijectively correspond to n-index subgroups of $\pi_1(\mathcal{M})$

• its n-fold coverings up to equivalence bijectively correspond to n-index subgroups of $\pi_1(\mathcal{M})$ up to conjugacy.

Non-conjugate subgroups vs epimorphisms onto C_m

$N_G(n)$:=#(conjugacy classes of n-index subgroups of group G).

Theorem [AM, 2006]. For any finitely generated group G,

$$\boxed{N_G(n) = \frac{1}{n} \sum_{\substack{m|n \\ mk=n}} \sum_{H \underset{k}{<} G} |\mathrm{Epi}(H, C_m)|}$$

$H \underset{k}{<} G$ denotes summing over subgroups of index k,
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- Clarifies everything! Easily calculated for $\pi_1(\mathcal{S})$, etc.

Lemma [G.Jones, 1995]. If $|\mathrm{Hom}(H, C_m)|$:=#(homomorphisms),

$$|\mathrm{Epi}(H, C_n)| = \sum_{d|n} \mu\Big(\frac{n}{d}\Big) |\mathrm{Hom}(H, C_d)|$$

Corollary. For a free group F_r, $|\mathrm{Epi}(F_r, C_n)| = \sum_{d|n} \mu\big(\frac{n}{d}\big) d^r$.

Jordan totient functions: definition

$\phi_k(n)$:=#(k-tuples jointly coprime to n).
Denoted often $J_k(n)$.

$$\phi_k(n) := n^k \prod_{p|n \text{ prime}} (1 - p^{-k}), \quad k \geq 0$$

• Defined as a multiplicative function:

$$\phi_k(p^a) := p^{k(a-1)}(p^k - 1), \quad p \text{ prime}, \ a \geq 1$$

• In particular, $\phi_1 = \phi$ (Euler totient).
In general, $\phi(n) | \ \phi_k(n), \ k \geq 1$.

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In general, $\phi(n) | \ \phi_k(n), \ k \geq 1$.

Proposition.

$$\phi_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k$$

- Easy. Is sometimes used as the definition of $\phi_k(n)$.

Jordan totient functions: free groups and coverings, revisited

Recall for the conjugacy classes of subgroups of F_r:

$$N_{F_r}(n) = \frac{1}{n}\sum_{m|n} M_{F_r}(m) \sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(r-1)m+1}$$

- As we just saw: $\sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(r-1)m+1} = \phi_{(r-1)m+1}\left(\frac{n}{m}\right)$.

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– the very first enumeration result with Jordan's functions.

• Such a considerable rôle of the Jordan function in this context has been realized (a simple observation) only recently [VL, 2003]. Rather popular presently.

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• Similarly: non-equivalent smooth coverings $N_{\mathcal{S}_\gamma}(n)$, etc.

Jordan totient functions: variations

Modified "odd" Jordan totient function:

$$\phi_k^{\text{odd}}(n) := n^k \prod_{\substack{p|n \\ p \text{ odd prime}}} (1 - p^{-k}), \quad k \geq 0$$

- Defined as a multiplicative function:

$$\phi_k^{\text{odd}}(p^a) := \phi_k(p^a),\ p \text{ odd}; \quad \phi_k^{\text{odd}}(2^a) := 2^{ka}$$

(For comparison: $\phi_k(2^a) = 2^{k(a-1)}(2^k - 1)$.)

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- $\phi_k^{\mathsf{odd}}(n)$ arises often in counting maps/coverings on non-orientable surfaces, or up to reflection [RN–AM &Co, 2008], or so-called circular maps [M.Deryagina, 2013].

- Also $\phi_k^{\mathsf{even}}(n)$ . . . Less significant.

Jordan functions: subgroups of F_2 and dessins d'enfants

$r := 2$. Free gr. F_2. Recall: $N_{F_2}(n) = \frac{1}{n} \sum_{m|n} \phi_{m+1}\left(\frac{n}{m}\right) M_{F_2}(m),\ n \geq 1.$

$N_{F_2}(n)$=#(transitive pairs of permutations up to joint conjugacy).
=#(non-isomorphic dessins d'enfants (aka hypermaps)).

• The initial numerical values: 1, 3, **7**, 26, 97, 624, 4163,. . .
(A057005 in the *On-Line Encyclopedia of Integer Sequences*).

.

Jordan functions: subgroups of F_2 and dessins d'enfants

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- "7" for $n = 3$ edges (in a bipartite representation):

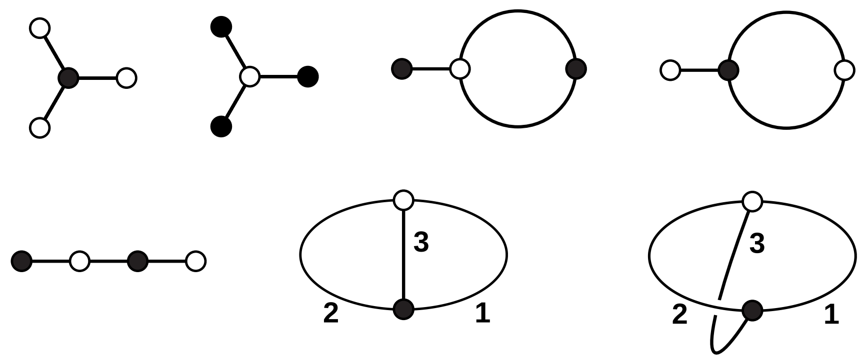

Figure 1. The dessins with three edges. The cyclic ordering at each vertex is indicated geometrically. The last two are distinct because of different cyclic orders at the bottom vertex—$(1, 3, 2)$ against $(1, 2, 3)$. [Pic.: L.Zapponi, Not. AMS, v.50, 2003]

Dedekind totient function: cyclic regular dessins d'enfants

$\psi(n)$. Related to Jordan functions: $\psi(n) := \frac{\phi_2(n)}{\phi(n)}$. Equivalently:

$$\psi(n) := n \prod_{p|n \text{ prime}} (1 + p^{-1})$$

- Defined as a multiplicative function:

$$\psi(p^a) := p^{a-1}(p+1), \quad p \text{ prime}, \ a \geq 1$$

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- A dessin [d'enfant] $\mathcal{D}$ is called cyclic regular if the group $\mathrm{Aut}(\mathcal{D})$ is cyclic and acts regularly on the edges.

$R(n)$:=#(non-isomorphic cyclic regular dessins with n edges).

Theorem [R.Nedela & Co, 2014].

$$R(n) = \psi(n)$$

- The very first appearance of Dedekind's psi in this context.

Sum/number of divisors: coverings of the torus/Klein bottle

Two more multiplicative functions. Exclusive applications.

Theorem. [AM, 1988]. For unrooted (connected) coverings of the (2-dim) torus $\mathcal{T}$ and the Klein bottle $\mathcal{K}$:

$$N_{\mathcal{T}}(n) = \sigma(n)$$

$$N_{\mathcal{K}}(n) = d(n) \text{ if } n \text{ is odd}$$

$$N_{\mathcal{K}}(n) = (5d(n/2) + \sigma(n/2))/2 \text{ if } n \equiv 2 \pmod 4$$

$$N_{\mathcal{K}}(n) = \ldots \text{ if } 4|n$$

where

$$\sigma(n) := \sum_{d|n} d \text{ (the sum of divisors)}$$

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Generalizations: $d^{\text{odd}}(n)$, ... in the same context.

II. A NEW MULTIVARIATE MULTIPLICATIVE FUNCTION

Return to unrooted planar maps. Loosely speaking:

$$A_0^+(n) = \frac{1}{2n}\Big(A_0'(n) + \sum_{\ell \geq 2,\, \ell | 2n} \phi(\ell)\widehat{A}_0'(n/\ell)\Big)$$

$A_0^+(n) :=$ #(arbitrary non-isomorphic n-edge planar maps),
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- What further for the torus and surfaces of greater genera?

The answer expected long ago: a similar reduction to rooted maps, i.e. a summation formula in terms of
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With respect to all possible finite automorphisms of the surface.

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• What further for the torus and surfaces of greater genera?

The answer expected long ago: a similar reduction to rooted maps, i.e. a summation formula in terms of
#(rooted quotient maps) with some coefficients.
With respect to all possible finite automorphisms of the surface.

• However: Which terms?? Which coefficients??

Non-planar maps: basic enumeration theorem

A far-reaching generalization of the planar case!

$\mathcal{S}_\gamma$: a closed oriented surface of genus $\gamma \geq 0$.
$A^+_\gamma(n)$=#(arbitrary unrooted maps with n edges on $\mathcal{S}_\gamma$).

Theorem (A.Mednykh–R.Nedela, 2006). (Loosely)

$$\boxed{A^+_\gamma(n) = \frac{1}{2n}\sum_{\ell|2n}\sum_{\substack{\Omega\in\mathrm{Orb}(\mathcal{S}_\gamma/C_\ell)\\ \Omega=\Omega(g;m_1,\ldots,m_r)}} |\mathrm{Epi_o}(\pi_1(\Omega),C_\ell)| \sum \#(\text{rooted q.m.})\ldots}$$

• $\mathrm{Orb}(\mathcal{S}_\gamma/C_\ell)$: the set of all cyclic orbifolds: quotient spaces by orientation-preserving actions of the cyclic group C_ℓ on $\mathcal{S}_\gamma$.

• Orbifold $\Omega = \Omega(g; m_1, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma/C_\ell)$: a closed surface with a distinguished finite set of branch points of *signature* $(g; m_1, \ldots, m_r)$, where g:= its genus,
m_j:= the orders of branch points.

Degression on rooted quotient maps

#(rooted quotient maps) in the RHS.

- Classes of generalized maps: with pendant semi-edges, etc.

- Easily reduce to ordinary rooted maps on orientable surfaces:
$A'_\delta(n),\ \delta \le \gamma$.

- Counted long ago with rather heavy formulae:
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• Counted long ago with rather heavy formulae: T.Walsh, A.Giorgetti.

• A great progress presently: much more efficient formulae, numerical results,... Outside of our topic.

Order preserving epimorphisms

$$A_\gamma^+(n) = \frac{1}{2n}\sum_{\ell|2n}\sum_{\substack{\Omega\in\mathrm{Orb}(\mathcal{S}_\gamma/C_\ell)\\ \Omega=\Omega(g;m_1,\ldots,m_r)}} |\mathrm{Epi_o}(\pi_1(\Omega),C_\ell)|\sum \#(\text{rooted q.m.})$$

$\mathrm{Epi_o}(\pi_1(\Omega),C_\ell)$:={order preserving epimorphisms $\pi_1(\Omega)\to C_\ell$}.

The fundamental group:

$$\pi_1(\Omega) = \Big\langle x_1,y_1,\ldots,x_g,y_g,z_1,\ldots,z_r\ : \prod_{i=1}^{g}[x_i,y_i]\prod_{j=1}^{r} z_j = 1,\quad z_j^{m_j}=1,\ j=1,\ldots,r\Big\rangle$$

• Order preserving epimorphism $\pi_1(\Omega)\to C_\ell$:
preserves the orders of the periodical generators z_j, $j=1,\ldots,r$
(aka: smooth epimorphism, or
epimorphism with the torsion-free kernel).

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• $|\mathrm{Epi_o}(\pi_1(\Omega),C_\ell)|$=??

Definition of the function $E(m_1, \ldots, m_r)$

Theorem [AM–RN, 2006]. For
$\Omega = \Omega(g; m_1, \ldots, m_r) \in \mathrm{Orb}(\mathcal{S}_\gamma / C_\ell)$,

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- $E(m_1,\ldots,m_r)$:= the number of solutions of the system

$$\left.\begin{array}{rl} x_1+\cdots+x_r \equiv & 0 \pmod{\ell} \\ \mathrm{GCD}(x_1,\ell) = & \ell/m_1 \\ \cdots & \\ \mathrm{GCD}(x_r,\ell) = & \ell/m_r \end{array}\right\}$$

where $\mathrm{LCM}(m_1,\ldots,m_r) = m$ and $m|\ell$.
$E(m_1,\ldots,m_r)$ does not depend on ℓ.

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where $\mathrm{LCM}(m_1, \ldots, m_r) = m$ and $m|\ell$.
$E(m_1, \ldots, m_r)$ does not depend on ℓ.

- Later I investigated this function and suggested to call it $\boxed{\text{orbicyclic}}$. It is a multivariate generalization of the Euler totient function.

Orbicyclic function: initial formula and elementary properties

Proposition [AM–RN]. For $m := \mathrm{LCM}(m_1, \ldots, m_r)$,

$$E(m_1, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} c_{m_1}(k) \cdots c_{m_r}(k)$$

$E(\emptyset) := 1 \qquad (m := 1 \text{ for } r = 0)$
where $c_n(k)$ is the famous Ramanujan (trigonometric) sum:

$$c_n(k) := \sum_{\substack{d\,(\mathrm{mod}\ n) \\ \mathrm{GCD}(d,n)=1}} \exp\left(\frac{2\,i\,k\,d}{n}\right)$$

Inconvenient for calculations and study.

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Inconvenient for calculations and study.

- $E(m_1, \ldots, m_r)$ is symmetric.
- $m_i = 1$ play no role. $(m_1, \ldots, m_r)$ is called reduced if all $m_j > 1$.
- $E(m) = 0$ for $m > 1$. $E(m_1, m_2) = 0$ for $m_1 \neq m_2$.
- $\boxed{E(m, m) = \phi(m)}$ – just the coefficients in our formula for counting planar maps!

Properties of the Ramanujan sum

Hölder's formula:

$$c_n(k) = \frac{\phi(n)}{\phi\left(\frac{n}{\mathrm{GCD}(k,n)}\right)}\,\mu\left(\frac{n}{\mathrm{GCD}(k,n)}\right)$$

- ϕ and μ again. Ramanujan's identity:

$$c_n(k) = \sum_{d|\mathrm{GCD}(k,n)} d\,\mu\left(\frac{k}{d}\right)$$

Lemma. $c_n(k)$ is multiplicative by n and for any p prime and $a \geq 1$,

$$c_{p^a}(k) = \begin{cases} (p-1)p^{a-1} & \text{if} \quad p^a|k \\ -p^{a-1} & \text{if} \quad p^a \nmid k,\ p^{a-1}|k \\ 0 & \text{otherwise} \end{cases}$$

- $c_n(k)$ is alternating. Unlike $E(m_1, \ldots, m_r)$.

Multivariate multiplicativity

A multivariate function $g = g(m_1, \ldots, m_r)$ is called multiplicative if $g(1, \ldots, 1) = 1$ and

$$g(m_1, \ldots, m_r) = g(m'_1, \ldots, m'_r) \cdot g(m''_1, \ldots, m''_r)$$

whenever $\quad m_j = m'_j m''_j, \quad j = 1, \ldots, r$, and $\mathrm{GCD}(M', M'') = 1$, where $M' = \prod_j m'_j$ and $M'' = \prod_j m''_j$.

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Theorem [VL, 2010].
The orbicyclic function $E(m_1, \ldots, m_r)$ is multiplicative.

• Thus: reduction to prime powers, i.e.
$E(m_1, \ldots, m_r)$ is determined by its values when $m = \text{GCD}(m_1, \ldots, m_r)$ is a prime power.

The primary case $m = p^a$. Main efficient formula

Introduce three parameters: $r = r_p$, $s = s_p$, $v = v_p$.

For $m_j = p^{a_j}$, $a_j > 0$, $j = 1, \dots, r$, without loss of generality,

$$a_1 = \ldots = a_{\boxed{s}} = a > a_{s+1} \cdots \geq a_{\boxed{r}} > 0.$$

$s :=$ <u>the multiplicity of the greatest exponent</u>.

$$v := \sum_{j \geq 2} (a_j - 1) = \sum_{j \geq 1} a_j - r - a + 1$$

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Theorem [VL, 2010].

$$\boxed{E(p^{a_1}, \ldots, p^{a_r}) = (p-1)^{r-s+1} p^v h_s(p)}$$

$h_s(x)$ is the (“*chromatic*”) polynomial $h_s(x) := \frac{(x-1)^{s-1} + (-1)^s}{x}$.

$$\begin{aligned}
h_1(x) &= 0 \\
h_2(x) &= 1 \\
h_3(x) &= x - 2 \\
h_4(x) &= x^2 - 3x + 3 \\
h_5(x) &= (x-2)(x^2 - 2x + 2)
\end{aligned}$$

Further properties

- $E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$ is non-negative and integer.

- $E(p^{a_1}, p^{a_2}, \ldots, p^{a_r})$ vanishes iff $s = 1$ or $p = 2$ and s is odd.

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- $\phi(m) | E(m_1, \ldots, m_r)$.

- $f_r(n) := E(\underbrace{n, \ldots, n}_{r})$ (the diagonal) is

the multiplicative function determined by

$$f_r(p^a) = (p-1)p^{(r-1)(a-1)} h_r(p), \quad p \text{ prime}, \ a \geq 1.$$

One more generalization of the Euler totient function:
in particular $f_2(n) = \phi(n)$.

I called $f_r(n)$ (by certain historical reasons)
the Rademacher–Brauer totient.

Non-vanishing conditions for $E(m_1, \ldots, m_r)$

Recall: $|\mathrm{Epi}_\mathrm{o}(\pi_1(\Omega), C_\ell)| = m^{2g} \cdot \phi_{2g}(\ell/m) \cdot E(m_1, \ldots, m_r)$

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and both $\phi_{2g}(\ell/m)$ and $E(m_1, \dots, m_r)$ do not vanish.

2. $\phi_{2g}(\ell/m) = 0$ iff

(e1) $m \nmid \ell$ [by definition, any $f(n) := 0$ for a non-integer argument] or

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2. $\phi_{2g}(\ell/m) = 0$ iff
(e1) $m \nmid \ell$ [by definition, any $f(n) := 0$ for a non-integer argument] or
(e2) $g = 0$ and $\ell > m$.

3. $E(m_1, \dots, m_r) = 0$ iff
(e3) $s_p = 1$ for some odd prime $p | m$ or
(e4) $2 | m$ and s_2 is odd. [s_2:= the multiplicity of the highest 2-power]

• (e3) & (e4) are equivalent to Harvey's conditions (1966) on branching data of finite cyclic groups acting on Riemann surfaces. Unexpected enumerative refinement.

Toth's summation formula

A compact number-theoretic representation:

Theorem [László Tóth, 2011].

$$E(m_1, \ldots, m_r) = \sum_{d_1|m_1, \ldots, d_r|m_r} \frac{d_1 \cdots d_r}{\mathrm{LCM}(d_1, \ldots, d_r)} \mu\Big(\frac{m_1}{d_1}\Big) \cdots \mu\Big(\frac{m_r}{d_r}\Big).$$

Further generalizations . . .

[L.Tóth, 2014]: an explicit formula for the generalized average

$$E_s = E_s(m_1, \ldots, m_r) := \frac{1}{m^{s+1}} \sum_{k=1}^{m} k^s c_{m_1}(k) \cdots c_{m_r}(k)$$

(so that $E_0 = E$).

Concluding remark: what are (generalized) totients?

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• **Tentative definition.** An arithmetic function $f(n)$ is called a totient iff it is multiplicative and the values

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• Suggested (in an equivalent form) by R.Vaidyanathaswamy in the classical paper [The theory of multiplicative arithmetic functions, *Trans. AMS*, 1931]. All totient functionds are such!

• It makes sense to restrict this definition: non-completely multiplicative functions with non-negative integer values . . .